

From classical chaos to decoherence in Robertson-Walker cosmology

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Abstract

We analyse the relationship between classical chaos and particle creation in Robertson-Walker cosmological models with gravity coupled to a scalar field. Within our class of models chaos and particle production are seen to arise in the same cases. Particle production is viewed as the seed of decoherence, which both enables the quantum to classical transition, and ensures that the correspondence between the quantum and classically chaotic models will be valid.

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Historically, physics has provided a description of the arrow of time by means of the second law of thermodynamics: The entropy of the Universe, or any isolated system, grows with time. However, physicists generally accept that this second law is not a fundamental law of nature; local, microphysical laws are invariant under time inversions, while temporal asymmetry manifests itself in macroscopic physics and arises from a restriction on the boundary conditions that selects the state of the universe among those that satisfy the dynamical laws [1]. In fact, a number of apparently different, well known arrows of time can be identified in different areas of physics, which are not completely independent. In this paper we would like to propose one such relationship, between the dynamical instabilities that act as seeds of chaos at the classical level, and the quantum to classical transition coming from particle creation in a cosmological scenario [2].

In a classical theory already, chaos opens up ways of explaining the origin of the arrow of time [3], and therefore, in a cosmological setting, of improving our knowledge of cosmological statistical mechanics and thermodynamics. Furthermore, just as non-integrability and dynamical instability are, at the classical level, the causes of chaos, they are also, at the quantum level, a cause of the instability of quantum states and particles. The question therefore arises, whether classical chaos and semiclassical particle production are related. At the semiclassical level, one can prove that decoherence and correlations, which cause the transition to the classical regime, arise in the presence of dynamical instability [4]; stable and unstable scalar field quanta are created during this process. The study of this connection between particle creation and chaos is therefore of great interest. If decoherence does induce a transition from quantum to classical, then it should be possible to utilize it in the context of quantum chaos to establish a more straightforward correspondence between the behavior of classically chaotic systems and their quantum counterparts [5].

The definition of chaos for quantum systems and its experimental consequences are still unclear. In previous works this study has been performed in different ways. Cooper et al. [6] considered the coupling between a quantum oscillator and a classical one, which leads a chaotic dynamical system when a critical value of the coupling and energy is reached. This

model could be related to the zero momentum part of the problem of pair production of charged scalar particles by a strong external electric field, and it would contain interesting information about the coupling between classical and quantum systems. Zurek and Paz [7] have shown the implications of the decoherence process for quantum chaos in a quantum open system composed of a chaotic system coupled to an environment. Decoherence destroys quantum interference, so the correspondence between quantum and classical models is maintained. The connection between classical and quantum systems have been also discussed by Cooper et al. [8] analyzing the relationship between chaos and the variational approximation.

With this goal in mind, we will study a cosmological metric coupled to a scalar field, and show that the conditions that produce chaotic behaviour at the classical level give semiclassical particle production. Particle creation is directly related to decoherence, and we will comment in our conclusions about the fact that decoherence is the key element in both the particle production \leftrightarrow classical chaos correspondence, and the quantum \rightarrow classical transition in our cosmological model studied here. Although the relation between classical chaos and decoherence could be very interesting to be raised in general cases (outside of our cosmological model studied at present), we will only refer to the Robertson-Walker cosmological case in the paper; new and more general results will be communicate in turn [9].

Let us consider a Robertson-Walker (RW) universe in conformal time gauge,

$$ds^2 = a^2(t)[-dt^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)] , \quad (1)$$

with dynamics described by the Einstein-Hilbert action

$$S_{\text{grav}}[g] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R , \quad (2)$$

conformally coupled to a real scalar field of mass μ , with action given by

$$S_{\text{matter}}[\Phi, g] = -\frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi + (\mu^2 + \frac{1}{6} R) \Phi^2 \right] . \quad (3)$$

If we reparametrize the scalar field by $\Phi \mapsto \phi = \sqrt{4\pi G/3} a\Phi$, we obtain from the above action and the metric in (1) the Hamiltonian

$$H(a, \phi; \pi, p) = \frac{1}{2} [-(\pi^2 + a^2) + (p^2 + \phi^2) + \mu^2 a^2 \phi^2] . \quad (4)$$

The system described by this Hamiltonian consists of two harmonic oscillators (of which one is “inverted,” as is to be expected from any degree of freedom related to the spatial volume element), coupled through a term proportional to μ^2 . For $\mu^2 = 0$, the system is trivially integrable; for nonvanishing μ^2 , the coupling term introduces chaos in the model, as shown in previous numerical [10] and analytical [11] work.

Let us summarize briefly our earlier results on the classical chaos in this model [11]. To begin with, we may replace p and ϕ by new dynamical variables j and φ , respectively, defined by

$$\phi = \sqrt{\frac{2j}{\omega}} \sin \varphi , \quad p = \sqrt{2\omega j} \cos \varphi , \quad (5)$$

where $\omega = \sqrt{1 + \mu^2 a^2}$ is the instantaneous frequency of the field. The Hamiltonian in terms of the new variables can be written as a sum $H = \hat{H}_0 + \delta \hat{H}$, of an unperturbed Hamiltonian which is obviously integrable, and a perturbation. With a second canonical transformation we may resolve the unperturbed dynamics [11]. We indicate the transformation by $(a, \varphi; P, j) \mapsto (\theta, \delta; k, j)$, where

$$k = \frac{j - h_0}{\sqrt{1 - \mu^2 j}} , \quad (6)$$

and the angle variables canonically conjugate to k and j , respectively, are

$$\theta = \arctan \left(\sqrt{1 - \mu^2 j} \, a/P \right) , \quad \delta = \varphi - \frac{\mu^2 a P}{4(1 - \mu^2 j)} , \quad (7)$$

the Hamiltonian $H = H_0 + \delta H$ can be written (for small μ^2) as

$$H_0(k, j) = j - k \sqrt{1 - \mu^2 j} , \quad (8)$$

and

$$\begin{aligned}
\delta H = & \frac{1}{4} \epsilon k j \left[\cos(2\theta + 2\delta) - \cos(2\theta - 2\delta) \right] + \frac{1}{4} \epsilon^2 \left[-\frac{3}{2} k^2 j - \frac{1}{4} k j^2 + \right. \\
& + (2k^2 j + \frac{1}{4} k j^2) \cos 2\theta - \frac{1}{2} k^2 j \cos 4\theta - \frac{1}{2} k^2 j \cos(2\delta - \pi/2) + \\
& + \frac{1}{4} k j^2 \cos 4\delta + k^2 j \cos(2\theta - 2\delta) - k^2 j \cos(2\theta + 2\delta) + \\
& - \frac{1}{8} k j^2 \cos(2\theta + 4\delta) - \frac{1}{8} k j^2 \cos(2\theta - 4\delta) + \\
& \left. + \frac{\sqrt{5}}{4} k^2 j \cos(4\theta + 2\delta + \psi) - \frac{\sqrt{5}}{4} k^2 j \cos(4\theta - 2\delta + \psi) \right] + O(\epsilon^3) , \tag{9}
\end{aligned}$$

where $\psi = -\arcsin(1/\sqrt{5})$, and the perturbation parameter ϵ is to be identified with μ^2 [11]. The dynamics of the unperturbed Hamiltonian is trivial, and gives a conditionally periodic motion $\theta = \theta_0 + \omega_k t$; $\delta = \delta_0 + \omega_j t$, with frequencies

$$\omega_k = \frac{\partial H_0}{\partial k} = -\sqrt{1 - \mu^2 j} , \quad \omega_j = \frac{\partial H_0}{\partial j} = \frac{\mu^2 k + 2\sqrt{1 - \mu^2 j}}{2\sqrt{1 - \mu^2 j}} . \tag{10}$$

The effect of the perturbation δH is felt in particular on the rational tori of the unperturbed dynamics, the ones where the motion becomes periodic, because the frequencies associated with the two degrees of freedom are related by the resonance condition

$$n_0 \omega_k + m_0 \omega_j = 0 , \tag{11}$$

for some integer numbers n_0 and m_0 . A calculation shows that the leading order perturbation in the Fourier expansion (9) for δH which is resonant with a torus in the $H = 0$ constraint surface is the second order term in ϵ with $(n_0, m_0) = (4, 2)$. Following the standard procedure to find how V_{42} affects the local dynamics near the resonant torus, we go over to a set of adapted canonical variables $(\gamma, \delta; K, J)$, chosen so that one of the momenta will still be a constant of the motion under the resonant perturbation:

$$K = \frac{k - k_0}{n_0} = \frac{1}{4} (k - k_0) , \tag{12}$$

$$\gamma = n_0 \theta + m_0 \delta + \psi_0 = 4 \theta + 2 \delta + \psi_0 , \tag{13}$$

$$J = -\frac{m_0}{n_0} k + j = -\frac{1}{2} k + j , \tag{14}$$

where ψ_0 is some arbitrary, fixed angle.

If we suppose that $K \ll 1$ is a small increment of the variable k around the resonant value k_0 , then we can expand the Hamiltonian, written in the new variables, in powers of K , and study the dynamics generated by the leading terms. We begin with the resonant part of the perturbation,

$$\begin{aligned} \delta H^{(2)} &= \epsilon^2 V_{42}^{(2)}(k, j) \cos(4\theta + 2\delta + \psi) + (\text{terms with different } n \text{ and } m) \\ &= \frac{\sqrt{5}}{16} \epsilon^2 k_0^2 j \cos(\gamma + \psi - \psi_0) + (\text{terms with different } n \text{ and } m) . \end{aligned} \quad (15)$$

Here, j is to be thought of as $j(K = 0, J)$, with J arbitrary. The perturbation depends only on γ and not on δ , J is still a constant of the motion; for simplicity we fix its value at the resonant one, J_0 . Then for the unperturbed Hamiltonian we obtain

$$H_0(K, J_0) = H_0(k_0, j_0) + \frac{1}{2} \tilde{\Omega} K^2 + O(K^3) , \quad (16)$$

where $\tilde{\Omega} = 8\mu^2 (1 - \mu^2 j_0)^{-1/2} + \mu^4 k_0 (1 - \mu^2 j_0)^{-3/2}$. So, to lowest order in K and ϵ , the K dynamics near the resonant torus is generated by

$$H_{\text{loc}}(\gamma, K) = H_0(k_0, j_0) + \frac{1}{2} \tilde{\Omega} K^2 + \frac{\sqrt{5}}{16} \epsilon^2 k_0^2 j_0 \cos(\gamma + \psi - \psi_0) , \quad (17)$$

which is the Hamiltonian of a well-known system, the non-linear pendulum.

This system has homoclinic orbits, and it can be shown [11] by means of the Melnikov method that a stochastic layer forms in the vicinity of the destroyed separatrix, which acts as a seed for chaos. Work is in progress on the calculation of the time scale for chaos to set in; this is an important aspect, and will be discussed elsewhere.

To show the quantum manifestation of the instabilities that lead to the onset of chaos in the classical model as described above, we will now consider the corresponding semiclassical model, in which Φ is a massive quantum scalar field on a classical curved background. Integrating out the quantum field, we will evaluate the closed time path (CTP) effective action, which provides us with the non-equilibrium effects of quantum fluctuations over the classical metric [12], and can be written as

$$e^{iS_{\text{eff}}[g^+, g^-]} = N e^{i(S_{\text{grav}}[g^+] - S_{\text{grav}}[g^-])} \int \mathcal{D}\Phi^+ \mathcal{D}\Phi^- e^{i(S_{\text{matter}}[\Phi^+, g^+] - S_{\text{matter}}[\Phi^-, g^-])}. \quad (18)$$

From this effective action it is possible to obtain the real and causal equation of motion, taking the functional variation of the action with respect to the $g_{\mu\nu}^+$ metric, and then setting $g_{\mu\nu}^+ = g_{\mu\nu}^-$. This CTP effective action has been explicitly evaluated [13], using the expression of the Euclidean effective action and the running of the coupling constants.

The CTP effective action is directly associated with the Decoherence Functional of Gell-Mann and Hartle [14], and can alternatively be written in terms of the Bogolubov coefficients connecting the in and out basis in each temporal branch. This fact implies that there is decoherence if and only if there is particle creation during the field evolution [4]. Explicitly, integrating the right-hand side of Eq. (18), we can write it as the Decoherence Functional

$$\mathcal{D}[g^+, g^-] = e^{i(S_{\text{grav}}[g^+] - S_{\text{grav}}[g^-] + \Gamma[g^+, g^-])}, \quad (19)$$

where Γ is the influence action for the scalar field, with the conformal factor being treated as an external field.

As the CTP effective action is independent of the out quantum state, we have the freedom of choosing an out particle model. It is convenient to choose a common out particle model for both evolutions (the Cauchy data are the same, although the actual basis functions will be different). The positive-frequency time dependent amplitude functions Φ_{\pm} for the conformal model in each branch are related to those F of the out model by $\Phi_{\pm} = \alpha_{\pm}F + \beta_{\pm}F^*$, where α_{\pm} and β_{\pm} are the Bogolubov coefficients in each temporal branch, obeying the normalization condition $|\alpha_{\pm}|^2 - |\beta_{\pm}|^2 = 1$. The CTP effective action in Eq. (19) is found to be [4]

$$\Gamma = \frac{i}{2} \ln[\alpha_- \alpha_+^* - \beta_- \beta_+^*]. \quad (20)$$

So, there is decoherence ($\text{Im } \Gamma > 0$) if and only if there is particle creation in different amounts in the two temporal evolutions.

The scalar fields on a Robertson-Walker metric can be separated into modes

$$\Phi(t, \vec{x}) = \sum_{\vec{k}} \Phi_{\vec{k}}(t) e^{i\vec{k} \cdot \vec{x}}, \quad (21)$$

where $\Phi_{\vec{k}}(t)$ are the amplitude functions of the \vec{k} -th mode. Defining $\phi_{\vec{k}}(t) := a(t)\Phi_{\vec{k}}(t)$ we can write down the wave equation for the \vec{k} -th mode,

$$\ddot{\phi}_{\vec{k}}(t) + \{k^2 + [\mu^2 + (\xi - \frac{1}{6}) R] a^2\} \phi_{\vec{k}}(t) = 0 . \quad (22)$$

(Notice that in terms of $\phi_{\vec{k}}$ the rescaled homogeneous field ϕ used in the classical model is $\phi = \sqrt{4\pi G/3} \phi_0$.) For massless conformally coupled ($\xi = \frac{1}{6}$) fields, $\phi_{\vec{k}}(t)$ admits the solutions

$$\phi_{\vec{k}}(t) = A e^{i\Omega t} + B e^{-i\Omega t} , \quad (23)$$

which are travelling waves in flat space. Since $\Omega = |\vec{k}| = \text{const}$, the positive and negative frequency components remain separated and there is no particle production. More generally, the wave equation for each mode has a time-dependent natural frequency given by

$$\Omega^2 = k^2 + [\mu^2 + (\xi - \frac{1}{6}) R] a^2 . \quad (24)$$

The dynamics of the background, through a and R , excites the negative frequency modes. This is the same problem as the Schrödinger equation with time-dependent potential $V(t) = [\mu^2 + (\xi - \frac{1}{6}) R] a^2$, which can induce backscattering of waves. In terms of this potential, we can write the Bogolubov coefficients up to lowest order in $V(t)$ as [15]

$$\begin{aligned} \alpha_k &= 1 + \frac{i}{2\Omega} \int_{-\infty}^{+\infty} V(t) dt \\ \beta_k &= -\frac{i}{2\Omega} \int_{-\infty}^{+\infty} V(t) e^{-i2\Omega t} dt . \end{aligned} \quad (25)$$

The number of created particles in the \vec{k} -th mode can alternatively be given in terms of $\dot{\phi}_{\vec{k}}$ and $\phi_{\vec{k}}$ by

$$N_{\vec{k}} = |\beta_{\vec{k}}|^2 = \frac{1}{2\Omega} \left[|\dot{\phi}_{\vec{k}}|^2 + \Omega^2 |\phi_{\vec{k}}|^2 \right] - \frac{1}{2} . \quad (26)$$

In this semiclassical model, we have evaluated the Bogolubov coefficients in a general mode decomposition. To make a real contact with the classical part described above, we must evaluate each expression in the homogeneous, $k = 0$ mode contribution; we have written the Bogolubov coefficients and the number of particles created in terms of different modes only to maintain generality in the semiclassical treatment.

In the massless case $V(t)$ is proportional to $(\xi - \frac{1}{6})$, and vanishes for conformal coupling. Therefore this term is present when the quantum fields are massive and/or when the coupling is not conformal. This is to be expected, since the imaginary part of the CTPEA is directly associated to gravitational particle creation. For massless, conformally coupled quantum fields, particle creation takes place only when spacetime is not conformally flat. Therefore the only contribution to the imaginary part of the CTPEA would be proportional to the square of the Weyl tensor in a general case. Since our present case is a conformally flat example, there is no particle creation when the conformally coupled field is massless. When the fields are massive and/or non-conformally coupled, particle creation takes place even if the Weyl tensor vanishes. This is why an additional contribution proportional to R^2 appears in the imaginary part of the effective action for the non conformal case [13].

We have shown that in a classical Robertson-Walker metric conformally coupled to a scalar matter field, the chaos appears when the field is massive. This is the same condition for particle creation to take place, at the semiclassical level. To make this point more explicit, we will show the “relation” between particle creation and the onset of chaos.

The Hamiltonian description of the cosmological particle creation is restricted to the dynamics of a finite system of parametric oscillators, where the Hamiltonian is simply [16]

$$\tilde{H}(t) = \frac{1}{2} \sum_{\vec{k}} (p^2 + \Omega^2 \phi_{\vec{k}}^2) = \sum_{\vec{k}} (N_{\vec{k}} + \frac{1}{2}) \Omega . \quad (27)$$

This is the field part of the Hamiltonian of Eq. (4) (there, it was evaluated in the $k = 0$ mode contribution). One can identify $|\phi_{\vec{k}}|^2$ and $|\dot{\phi}_{\vec{k}}|^2$ with the canonical coordinates ϕ^2 and p^2 respectively, the eigenvalue of \tilde{H} being the energy $\tilde{E} = (N_{\vec{k}} + \frac{1}{2}) \Omega$. When $\mu = 0$, the zero mode Hamiltonian $\tilde{H} = 0$, and the total Hamiltonian of Eq. (4) is integrable (an inverted harmonic oscillator Hamiltonian). Therefore the resonance condition Eq. (11) can be explicitly written in terms of the number of created particles as

$$\frac{m_0}{n_0} = \frac{2 [1 - \mu^2 \Omega_0 (N_0 + \frac{1}{2}) / \omega]^{3/2}}{2 - \mu^2 \Omega_0 (N_0 + \frac{1}{2}) / \omega} , \quad (28)$$

where $\Omega_0^2 = \mu^2 a^2$ and N_0 are the time dependent frequency and the particle number in the zero mode contribution, respectively. Therefore in the massless case (no particle creation)

this condition is only satisfied by $n_0 = m_0$ and $\omega_k = \omega_j = 1$, *i.e.*, these frequencies are completely degenerate. This agrees with what we saw in the classical model, where $\mu^2 = 0$ implies $\omega_k = \omega_j = 1$ from Eq. (10), so the perturbative treatment is not applicable, but there is no perturbation and chaos disappears in the massless case.

Decoherence is the key to understanding the relationship between the arrows of time in cosmology. In the context of quantum open systems, where the metric is viewed as the “system” and the quantum fields as the “environment,” decoherence is produced by the continuous interaction between system and environment. The non-symmetric transfer of information from system to environment is the origin of an entropy increase (in the sense of von Neumann), because there is loss of information in the system, and of the time asymmetry in cosmology, because growth of entropy, particle creation and isotropization show a tendency towards equilibrium. However, decoherence is also a necessary condition for the quantum to classical transition. In the density matrix formulation, decoherence appears as the destruction of interference terms and, in our model, as the transition from a pure to a mixed state in the time evolution of the density matrix associated with the RW metric; the interaction with the quantum modes of the scalar fields is the origin of such a non-unitary evolution [17].

In this paper we saw that, in the cosmological model we considered, particle creation and decoherence are the effect of resonances between the evolutions of the scale factor a and the free massive field ϕ , which is the origin of the chaotic behaviour. In the semiclassical treatment we used a general coupling ξ between the metric and the scalar field, and saw that a non-conformal coupling effectively led to a time-dependent mass $M^2 = \mu^2 + (\xi - \frac{1}{6}) R$ for ϕ , and particle production, even when $\mu^2 = 0$; a perturbative analysis of the classical model for $\xi \neq \frac{1}{6}$ would reveal the presence of chaos, again even in the massless case; a similar conclusion would be reached by analyzing a model with a $\lambda\phi^4$ self-interaction term in the action.

ACKNOWLEDGMENTS

This work was supported by Universidad de Buenos Aires, CONICET and Fundación Antorchas. We are grateful to D.F. Mazzitelli, G.B. Mindlin and J.P. Paz for many useful discussions.

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